1) $A, B \in M(n \times n / \mathbb{R})$

$\exists A^{-1}\ A B^{-1} \Rightarrow \exists (AB)^{-1}$

$A^{-1} = \dfrac{1}{\det A} (\det A_{i,j})^t \qquad \operatorname{adj} A = (\det A_{i,j})^t$

$\hookrightarrow A \cdot A^{-1} = \dfrac{1}{\det A} (\operatorname{adj} A) A \Rightarrow I = \operatorname{adj} A \dfrac{A}{\det A}$

$(\operatorname{adj} A)^{-1} = \dfrac{A}{\det A} \qquad (\operatorname{adj} B)^{-1} = \dfrac{B}{\det B}$

$(\operatorname{adj} AB)^{-1} = \dfrac{A \cdot B}{\det(AB)} = \dfrac{A}{\det A} \cdot \dfrac{B}{\det B} = (\operatorname{adj} A)^{-1} (\operatorname{adj} B)^{-1}$

sorı 3 + ham 4 ox. on!

2) $T : \mathbb{R}_2[x] \to M(2 \times 2, \mathbb{R})$

$T(f) = \begin{pmatrix} 2f'(0) & f(1) \\ f''(2) & 0 \end{pmatrix}$

$T(af + bg) = \begin{pmatrix} 2(af + bg)'(0) & (af + bg)(1) \\ (af + bg)''(2) & 0 \end{pmatrix}$

$= \begin{pmatrix} 2af'(0) + 2bg'(0) & af(1) + bg(1) \\ af''(2) + bg''(2) & 0 \end{pmatrix}$

$= \begin{pmatrix} 2af'(0) & af(1) \\ af''(2) & 0 \end{pmatrix} + \begin{pmatrix} 2bg'(0) & bg(1) \\ bg''(2) & 0 \end{pmatrix}$

$= a \begin{pmatrix} 2f'(0) & f(1) \\ f''(2) & 0 \end{pmatrix} + b \begin{pmatrix} 2g'(0) & g(1) \\ g''(2) & 0 \end{pmatrix} = aT(f) + bT(g)$

$$T(f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow f \in \ker T$$

$$\begin{pmatrix} 2f'(0) & f(1) \\ f''(2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$f(x) = \alpha x^2 + \beta x + \gamma$ $\qquad$ $f(1) = \alpha + \beta + \gamma$

$f'(x) = 2\alpha x + \beta$ $\qquad$ $f'(0) = \beta$

$f''(x) = 2\alpha$ $\qquad$ $f''(2) = 2\alpha$

$$\begin{pmatrix} 2\beta & \alpha+\beta+\gamma \\ 2\alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\beta = 0 \quad \alpha = 0 \quad \alpha+\beta+\gamma = 0 \Rightarrow \gamma = 0$

$f \in \ker T \Leftrightarrow f(x) = 0 \Leftrightarrow \ker T = \{0\} \Leftrightarrow T$ 1-1

$T(\mathbb{R}_2[x])$

$$T(f) = \begin{pmatrix} 2\beta & \alpha+\beta+\gamma \\ 2\alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 2\alpha & 0 \end{pmatrix} + \begin{pmatrix} 2\beta & \beta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$$

$$T(f) = \alpha \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \beta \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \alpha, \beta, \gamma \in \mathbb{R}$$

$$T(\mathbb{R}_2[x]) = \left\langle \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$$

Επειδή οι πίνακες $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ είναι γρ. ανεξάρτητοι

η διάσταση της εικόνας είναι 3

$$\underset{3}{\dim \mathbb{R}_2[x]} = \underset{0}{\dim \ker T} + \underset{3}{\dim T(\mathbb{R}_2[x])}$$

Η T δεν είναι επί (διάσταση $3 \neq 4$ → του πίνακα)

3) $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad T(x,y,z) = (y, z, -x-y-z)$

$\ker T$

$T(x,y,z) = (0,0,0) \Leftrightarrow (y, z, -x-y-z) = (0,0,0)$

$y = 0$

$z = 0$

$-x-y-z = 0 \quad x = 0 \quad \ker T = \{(0,0,0)\}$ ⇒ T είναι 1-1

$T: \mathbb{R}^{(3)} \to \mathbb{R}^{(3)}$ $\quad T$ είναι 1-1 → T επί

T είναι $\cong$

$T^{-1}: \mathbb{R}^3 \to \mathbb{R}^3$

$$(T, \text{κανονικές βάσεις}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

Θα βρατε τον $(T, \text{κανονικές βάσεις})^{-1}$ και θα τον πολ/με με $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$T^{-1}(x,y,z) = \left(\underbrace{(T, \text{κανονικές βάσεις})}_{\text{αντίστροφος}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right)^t$$

SOS

6) $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x,y,z) = (x+y+kz,\ x+ky+z,\ 2y+2z)$

$\dim \ker T = 1$

$T(x,y,z) = (0,0,0)$

$x+y+kz = 0 \to (1-k)y + y - ky = 0 \Leftrightarrow y(2-2k) = 0$

$z + x + ky = 0 \Rightarrow x + ky - y = 0 \Rightarrow x = (1-k)y$

$2z + 2y = 0 \Rightarrow z = -y$

$2y(1-k) = 0$

Αν $y = 0 \quad x = y = z = 0 \quad \ker T = \{(0,0,0)\}$ Αδύνατο

Αν $k = 1 \quad x = (1-k)y = 0 \quad z = -y$

$\ker T = \{(0, y, -y) \mid y \in \mathbb{R}\} = \{y(0,1,-1) \mid y \in \mathbb{R}\} = \langle (0,1,-1) \rangle \quad k = 1$

$T(x,y,z) = (x+y+z,\ x+y+z,\ 2y+2z)$

$\mathrm{Im}T = \langle T(1,0,0),\ T(0,1,0),\ T(0,0,1) \rangle$ (οποιεσδήποτε βάσεις)

$= \langle (1,1,0)(1,1,2),\ (1,1,2) \rangle$

$\Rightarrow \mathrm{Im}T \langle (1\,1\,0)(1\,1\,2) \rangle$ - βάση

$$(T, \underbrace{\text{κανονικές βάσεις}}_{\Sigma}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} = A$$

$(T, \underbrace{\text{βάση } (1,1,-1),(1,-1,0)(-1,0,0)}_{\Sigma'})$

$T(1,1,-1) = (1,1,0) = \alpha(1,1,-1) + \beta(1,-1,0) + \gamma(-1,0,0)$

$1 = \alpha + \beta - \gamma \qquad \gamma = -2$

$1 = \alpha - \beta \qquad \beta = -1$

$0 = -\alpha \Rightarrow \alpha = 0$

$T(1,-1,0) = (0,0,-2) = \kappa(1,1,-1) + \lambda(1,-1,0) + \mu(-1,0,0)$

$0 = \kappa + \lambda - \mu \qquad \mu = 4$

$0 = \kappa - \lambda \qquad \lambda = 2$

$-2 = -\kappa \Rightarrow \kappa = 2$

$T(-1,0,0) = (-1,-1,0)$

$(-1,-1,0) = \alpha'(1,1,-1) + \beta'(1,-1,0) + \gamma'(-1,0,0)$

$-1 = \alpha' + \beta' - \gamma' \qquad \gamma' = 2$

$-1 = \alpha' - \beta' \qquad \beta' = 1$

$0 = -\alpha' \qquad \alpha' = 0$

$$B = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 2 & 1 \\ -2 & 4 & 2 \end{pmatrix}$$

$B = P^{-1} A P$

$(T \Sigma' \Sigma') = \qquad (T, \Sigma, \Sigma)$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "T"] & \mathbb{R}^3 \\ \Sigma' \arrow[r, "B"'] \arrow[d, "P"] & \Sigma' \\ \Sigma \arrow[r, "A"'] & \Sigma \arrow[u, "P^{-1}"] \end{tikzcd}$$

Ζητάμε τον πίνακα P ο οποίος είναι πίνακας αλλαγής βάσης από τη Σ' στη Σ'

$(1,1,-1) = 1(1,0,0) + 1(0,1,0) - 1(0,0,1)$

$(1,-1,0) = 1(1,0,0) - 1(0,1,0) + 0(0,0,1)$

$(-1,0,0) = -1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Ο P^{-1} είναι πίνακας αλλαγής βάσης από τη Σ στη Σ'

$(1,0,0) = \alpha(1,1,-1) + \beta(1,-1,0) + \gamma(-1,0,0)$

$1 = \alpha + \beta - \gamma$

$0 = \alpha - \beta$

$0 = -\alpha \qquad \alpha = \beta = 0 \qquad \gamma = -1$

$(0,1,0) = \alpha'(1,1,-1) + \beta'(1,-1,0) + \gamma'(-1,0,0)$

$0 = \alpha' + \beta' - \gamma' \qquad \gamma' = -1$

$1 = \alpha' - \beta' \qquad \beta' = -1$

$0 = -\alpha' \qquad \alpha' = 0$

$(0,0,1) = \alpha''(1,1,-1) + \beta''(1,-1,0) + \gamma''(-1,0,0)$

$0 = \alpha'' + \beta'' - \gamma''$ $\quad\quad \gamma'' = -2$

$0 = \alpha'' - \beta''$ $\quad\quad \beta'' = -1$

$1 = -\alpha''$ $\quad\quad \alpha'' = -1$

$$P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & -2 \end{pmatrix}$$

7) $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad \varepsilon_1 = (1,1,1)$ $\quad \varepsilon_2 = (1,1,0)$ $\quad \varepsilon_3 = (1,0,0)$

$\Sigma = ((1,1,1),(1,1,0),(1,0,0))$

$$(T,\Sigma,\Sigma) = A = \begin{pmatrix} -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{3}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

$(x,y,z) = \alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,0)$

$x = \alpha + \beta + \gamma \Rightarrow \gamma = x - \alpha - \beta \Rightarrow \gamma = x - z - y + z \Rightarrow \boxed{\gamma = x - y}$

$y = \alpha + \beta \Rightarrow \beta = y - \alpha \Rightarrow \boxed{\beta = y - z}$

$\boxed{z = \alpha}$

$(x,y,z) = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

$T(1,1,1) = -\frac{1}{2}(1,1,1) + \frac{3}{2}(1,1,0) + \frac{5}{2}(1,0,0)$

$= \left(\frac{7}{2}, 1, -\frac{1}{2}\right)$

$T(1,1,0) = 2(1,1,1) + 0(1,1,0) - 1(1,0,0)$
$= (1,2,2)$

$T(1,0,0) = \frac{1}{2}(1,1,1) + \frac{1}{2}(1,1,0) + \frac{1}{2}(1,0,0)$

$= \left(\frac{3}{2}, 1, \frac{1}{2}\right)$

$$T(x,y,z) = z\,T(1,1,1) + (y-z)\,T(1,1,0) + (x-y)\,T(1,0,0)$$
$$= z\left(\frac{7}{2}, 1, -\frac{1}{2}\right) + (y-z)(1,2,2) + (x-y)\left(\frac{3}{2}, 1, \frac{1}{2}\right)$$

$$T(x,y,z) = \Big(\frac{7}{2}z + y - z + \frac{3}{2}x - \frac{3}{2}y,$$

$$z + 2y - 2z + x - y,$$
$$-\frac{z}{2} + 2y - 2z + \frac{x}{2} - \frac{y}{2}\Big)$$

$$= \left(\frac{3}{2}x - \frac{y}{2} + \frac{7}{2}^{5} z,\ x + y - z,\ \frac{x}{2} + \frac{3y}{2} - \frac{5}{2}z\right)$$

Να βρω

↓

$\ker T = \dots$

$\operatorname{Im} T = \dots$